<table>
<thead>
<tr>
<th>outline</th>
</tr>
</thead>
<tbody>
<tr>
<td>structure formation equations</td>
</tr>
</tbody>
</table>

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structure formation equations

### Cosmic Structure Formation

Structure formation is a self-gravitating, fluid mechanical phenomenon.

- **Continuity Equation**: Evolution of the density field due to fluxes
  \[
  \frac{\partial}{\partial t} \rho + \text{div}(\rho \vec{u}) = 0 \tag{1}
  \]

- **Euler Equation**: Evolution of the velocity field due to forces
  \[
  \frac{\partial}{\partial t} \vec{u} + \vec{u} \cdot \nabla \vec{u} = -\nabla \Phi \tag{2}
  \]

- **Poisson Equation**: Potential sourced by density field
  \[
  \Delta \Phi = 4\pi G \rho \tag{3}
  \]

- 3 quantities, 3 equations → solvable
- 2 nonlinearities: \(\rho \vec{u}\) in continuity and \(\vec{u} \cdot \nabla \vec{u}\) in Euler equation
viscosity and pressure

dynamics with dark matter

dark matter is collisionless (no viscosity and pressure) and interacts gravitationally (non-saturating force)

- dark matter is collisionless → no mechanism for microscopic elastic collisions between particles, only interaction by gravity
- derivation of the fluid mechanics equation from the Boltzmann-equation: moments method
  - continuity equation
  - Navier-Stokes equation
  - energy equation
- system of coupled differential equations, and closure relation
- effective description of collisions: viscosity and pressure, but
- relaxation of objects if there is no viscosity?
  - stabilization of objects against gravity if there is no pressure?
collective dynamics: dynamical friction

- dynamical friction emulates viscosity: there is no microscopic model for viscosity, but collective processes generate an effective viscosity
  - a particle moving through a cloud produces a wake
  - behind the particle, there is a density enhancement
  - density enhancement breaks down particle velocity
- kinetic energy of the incoming object is transformed to unordered random motion
Kelvin-Helmholtz instability

- shear flows become unstable if there are large perpendicular velocity gradients
- generation of vorticity in shear flows by the Kelvin-Helmholtz instability
- absent in the case of dark matter: flow is necessarily laminar
vorticity

- Intuitive explanation of the nonlinearity of the Navier-Stokes eqn
  \[
  \frac{\partial}{\partial t} \vec{u} + \vec{u} \nabla \vec{u} = \frac{\nabla p}{\rho} - \nabla \Phi + \mu \Delta \vec{u}
  \]  
  (4)

- Vorticity equation: \( \vec{\omega} \equiv \text{rot} \vec{u} \)
  \[
  \frac{\partial \vec{\omega}}{\partial t} + \vec{u} \nabla \vec{\omega} = \vec{\omega} \nabla \vec{u} - \vec{\omega} \text{div} \vec{u} + \frac{1}{\rho^2} \nabla p \times \nabla \rho + \mu \Delta \vec{\omega}
  \]
  (5)
  
  - Generation of vorticity by
    - Pressure gradients non-parallel to density gradients
    - Viscous stresses
  
  \( \rightarrow \) Not present in the case of collisionless dark matter
  \( \rightarrow \) Gravity as a conservative force is not able to induce vorticity
look at overdensity field $\delta \equiv (\rho - \bar{\rho})/\bar{\rho}$, with $\bar{\rho} = \Omega_m \rho_{\text{crit}}$

- analytical calculations are possible in the regime of linear structure formation, $\delta \ll 1$
  $\to$ homogeneous growth, dependence on dark energy, number density of objects

- transition to non-linear structure growth can be treated in perturbation theory (difficult!), $\delta \sim 1$
  $\to$ first numerical approaches (Zel’dovich approximation), directly solvable for geometrically simple cases (spherical collapse)

- non-linear structure formation at late times, $\delta > 1$
  $\to$ higher order perturbation theory (even more difficult), ultimately: direct simulation with n-body codes
linearisation: perturbation theory for $\delta \ll 1$

- transition from relativistic gravity on large to Newtonian gravity on small scales: really difficult conceptually, this is a pantomime moment if there ever was one
- move from physical to comoving frame, related by scale-factor $a$
- use density $\delta = \Delta \rho / \rho$ and comoving velocity $\vec{u} = \vec{v} / a$
  - linearised continuity equation:
    $$\frac{\partial}{\partial t} \delta + \text{div} \vec{u} = 0$$
  - linearised Euler equation: evolve momentum
    $$\frac{\partial}{\partial t} \vec{u} + 2H(a)\vec{u} = -\frac{\nabla \Phi}{a^2}$$
  - Poisson equation: generate potential
    $$\Delta \Phi = 4\pi G \rho_0 a^2 \delta$$
growth equation

- structure formation is homogeneous in the linear regime, all spatial derivatives drop out
- combine continuity, Jeans- and Poisson-eqn. for differential equation for the temporal evolution of $\delta$

$$\frac{d^2 \delta}{da^2} + \frac{1}{a} \left( 3 + \frac{d \ln H}{d \ln a} \right) \frac{d\delta}{da} = \frac{3\Omega_m(a)}{2a^2} \delta$$

- growth function $D_+(a) \equiv \delta(a)/\delta(a = 1)$ (growing mode)
  - position and time dependence separated: $\delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x})$
  - in Fourier-space modes grows independently: $\delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k})$
- for standard gravity, the growth function is determined by $H(a)$
terms in the growth equation

- two terms in growth equation:
  - source \( Q(a) = \Omega_m(a) \): large \( \Omega_m(a) \) make the gravitational fields strong
  - dissipation \( S(a) = 3 + \frac{d \ln H}{d \ln a} \): structures grow if their dynamical time scale is smaller than the Hubble time scale \( \frac{1}{H(a)} \)
\[ D_+(a) \text{ for } \Omega_m = 1 \text{ (dash-dotted), for } \Omega_\Lambda = 0.7 \text{ (solid) and } \Omega_k = 0.7 \text{ (dashed)} \]

- density field grows \( \propto a \) in \( \Omega_m = 1 \) universes, faster if \( w < 0 \)
mode coupling

- linear regime structure formation: homogeneous growth

\[ \delta(\vec{x}, a) = D_+(a)\delta_0(\vec{x}) \rightarrow \delta(\vec{k}, a) = D_+(a)\delta_0(\vec{k}) \]  \hspace{1cm} (7)

- separation fails if the growth is nonlinear, because a void can’t get more empty than \( \delta = -1 \), but a cluster can grow to \( \delta \simeq 200 \)

\[ \delta(\vec{x}, a) = D_+(a, \vec{x})\delta_0(\vec{x}) \]  \hspace{1cm} (8)

- product of two \( \vec{x} \)-dependent quantities in real space \( \rightarrow \) convolution in Fourier space:

\[ \delta(\vec{k}, a) = \int d^3k' D_+(a, \vec{k} - \vec{k}')\delta_0(\vec{k}') \]  \hspace{1cm} (9)

- \( k \)-modes do not evolve independently: mode coupling

- correlation produces a non-Gaussian field (central limit theorem)
perturbation theory

- perturbative series in density field:
  \[ \delta(\vec{x}, a) = D_+(a)\delta^{(1)}(\vec{x}) + D_+^2(a)\delta^{(2)}(\vec{x}) + D_+^3(a)\delta^{(3)}(\vec{x}) + \ldots \] (10)

- lowest order:
  \[ \delta^{(2)}(\vec{k}) = \int \frac{d^3p}{(2\pi)^3} M_2(\vec{k} - \vec{p}, \vec{p})\delta(\vec{p})\delta(|\vec{k} - \vec{p}|) \] (11)

- with mode coupling
  \[ M_2(\vec{p}, \vec{q}) = \frac{10}{7} + \frac{\vec{p}\vec{q}}{pq} \left( \frac{p}{q} + \frac{q}{p} \right) + \frac{4}{7} \left( \frac{\vec{p}\vec{q}}{pq} \right)^2 \] (12)

- properties:
  - time-independent, no scale \( \vec{p}_0 \)
  - strongest coupling if \( \vec{p} = \vec{q} \)
  - some coupling of modes \( \vec{p} \perp \vec{q} \)
  - no coupling if \( \vec{p} = -\vec{q} \)
homogeneity, linearity and Gaussianity

...almost the same thing in structure formation!

- **linearity**
  - eqns can be linearised: $|\delta| \ll 1$
  - linearisation fails: $|\delta| \approx 1$

- **homogeneity**
  - homogeneous: $\delta(\vec{x}, a) = D_+(a)\delta(\vec{x}, a = 1)$
  - inhomogeneous: $\delta(\vec{x}, a) = D_+(\vec{x}, a)\delta(\vec{x}, a = 1)$

- **Gaussianity** (with central limit theorem)
  - Gaussian amplitude distribution $p(\delta)d\delta$
  - non-Gaussian (lognormal) distribution $p(\delta)d\delta$

**mode coupling**

easiest way to visualise: resonance phenomenon
- linearity, homogeneity and Gaussianity imply each other
- nonlinear structure formation breaks homogeneity and produces non-Gaussian statistics
- mode coupling - can be described in perturbation theory
link between dynamics and statistics

- nonlinear structure formation couples modes
- superposition of various k-modes (not independent anymore) generate a non-Gaussian density field
- non-Gaussian density field:
  - odd moments are not necessarily zero
  - even moments are not powers of the variance
- finite correlation length: n-point correlation functions
  - 3-point-function: bispectrum
  - 4-point-function: trispectrum

higher order correlations quickly become unpractical, and are really difficult to determine
nonlinear CDM spectrum $P(k)$

- fit to numerical data, $z = 9, 4, 1, 0$, normalised on large scales
- extra power on large scales, time dependent, saturates
- on top of scaling $P(k, a) \propto D^2_+(a)$
n-body simulations of structure formation

- basic issue: gravity is long-ranged, for each particle the gravitational force of all other particle needs to be summed up, complexity $n^2$
- algorithmic challenge to break down $n^2$-scaling
  - particle-mesh
  - particle$^3$-mesh
  - tree-codes
  - tree-particle mesh
gravothermal instability: thermal energy

- consider self-gravitating system, exchanging (thermal) energy with environment, Lynden-Bell & Wood (1968)
- example: cluster of galaxies loses energy in the form of thermal X-ray radiation, Coma: few $10^{44}$ erg/sec
  1. energy is removed from a self-gravitating object, on a time-scale $t_{\text{remove}} \gg$ dynamical time-scale $t_{\text{dyn}}$
  2. system assumes a new equilibrium state deeper inside its own potential well (quasi-stationary, no relaxation)
  3. release of gravitational binding energy, particles speed up
  4. velocity dispersion (temperature) rises
- reacts on removal of thermal energy by heating up!
- self-gravitating systems have a negative specific heat $c$
- systems cool, if $t_{\text{remove}} \ll t_{\text{dyn}}$, in this case $c > 0$
- stability of self-gravitating non-isolated systems?
gravothermal instability: particles

- stars continuously reshuffle their kinetic energy in a globular cluster
- kinetic energy of a star fluctuates, can get gravitationally unbound
- star leaves cluster on parabolic orbit, does not take away energy
- gravitational binding energy distributed among fewer stars
- system heats up by evaporating stars, eventually disintegrates
• in the dynamical evolution, systems tend towards a final state which is not very sensitive on the initial conditions → relaxation

• usually, this is accompanied by generation of entropy, which defines an arrow of time

• in cosmology, galaxies with very similar properties form from a Gaussian fluctuation in the matter distribution

• but: dark matter is a collisionless fluid!
  • no viscosity in Euler-eqn. which can dissipate velocities
  • transformation from kinetic energy to heat is not possible
  • no Kelvin-Helmholtz instability and Kolmogorov cascading
  • Euler-equation is time-reversible and no entropy is generated
  • relaxation does not take place
relaxation: 1. two-body relaxation

Relaxation with **Keplerian** (time-reversible) orbits in a succession of two-body encounters

- Consider a system with $N$ stars of size $R$, density of stars is $n \sim N/R^3$, total mass $M = Nm$
- Shoot a single star into the cloud and track its transverse velocity
- In a single encounter, the velocity changes
  \[
  \delta u_\perp (\text{single}) \sim \frac{Gm}{b^2} \frac{2b}{u} \sim \frac{2Gm}{bu} \tag{13}
  \]
  with impact parameter $b$, using Born-approx. with $\delta t = 2b/u$
- Multiple encounters: add random kicks, so variance $\delta u^2_\perp$ grows
  \[
  \frac{d}{dt} \delta u^2_\perp \sim 2\pi \int bdb \delta u_\perp (\text{single}) n u = \frac{8\pi G^2 m^2 n}{u} \ln\left(\frac{b_{\text{max}}}{b_{\text{min}}}\right) \tag{14}
  \]

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Crash Course Cosmological Physics
relaxation: 2. dynamical friction

- system of reference with moving particle
- all other particle zoom past on hyperbolic orbits, orbit/gravitational scattering depends sensitively on the impact parameter
- directed, ordered velocities → random transverse velocities

source: J. Schombert
relaxation: 3. violent relaxation

• proposed by Lynden-Bell for explaining the brightness profile of elliptical galaxies, wipes out structure of spiral galaxies in the merging

• each particle sees a rapidly fluctuating potential generated by all particles

\[
\frac{dE}{dt} = m \frac{du^2}{dt} + \frac{\partial \Phi}{\partial t} + \vec{u} \nabla \Phi
\]  

(15)

• dynamic kind of scattering mediated by grav. field

with \[ \frac{du^2}{dt} = 2 \vec{u} \frac{d\vec{u}}{dt} = -\frac{2}{m} \vec{u} \nabla \Phi \]

\[ \rightarrow \quad \frac{dE}{dt} = \frac{\partial \Phi}{\partial t} \]  

(16)

• even particles with initially similar trajectories get separated
relaxation: 4. phase space mixing

globular cluster Palomar-5, source: J. Staude

- time evolution of a globular cluster orbiting the Milky Way:
  - stars closer to Galactic centre move faster
  - stars further away move slower
- with time, the streams get more elongated and eventually form a tightly wound spiral

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relaxation: 4. phase space mixing

- naive interpretation: system produces structure on smaller and smaller scales (spiral winds up), eventually crosses thermodynamic scale $\lambda$
- but: the system is time-reversible and does conserve full phase space information
- relaxation does not take place, the system remembers its initial conditions
- thermodynamic scale is not well defined, gravity is a power law!
- solution: no matter how small the thermodynamic scale is chosen, the system will always wipe out structures above this scale with time $\rightarrow$ coarse-graining

generation of entropy

phase space density $f$ measured above this scale decreases, and entropy $S \propto \int \! d^3p \! d^3q \! f \ln f$ increases
Navarro-Frenk-White profile
Navarro-Frenk-White profile

- Navarro, Frenk + White: haloes in n-body simulation show a profile:

\[ \rho \propto \frac{1}{x(1 + x^2)} \quad \text{with} \quad x \equiv \frac{r}{r_c} \quad \text{and} \quad r_c = c r_{\text{vir}} \]  

- universal density profile, applicable to haloes of all masses
- fitting formula breaks down:
  - infinite core density
  - total mass diverges logarithmically
- very long lived transitional state (gravothermal instability)
- scale radius \( r_s \) is related to virial radius by concentration parameter \( c \)
- \( c \) has a weak dependence on mass in dark energy models
galaxy biasing

GIF-simulation, Kaufmann et al.
galaxy bias models

- galaxies trace the distribution of dark matter
- simplest (local, linear, static, morphology and scale-indep.) relation:
  \[
  \frac{\delta n}{\langle n \rangle} = b \frac{\rho}{\langle \rho \rangle}
  \]  
  (18)

  with bias parameter \(b\)

- bias models:
  - massive objects are more clustered (larger \(b\)) than low-mass objects
  - red galaxies are stronger clustered than blue galaxies
  - bias is slowly time evolving and decreases

- physical explanation: galaxies form at local peaks in the dark matter field, and reflect the local matter density directly

- naturally: \(\xi_{\text{galaxy}}(r) = b^2 \xi_{\text{CDM}}(r)\) for the above model
stability of elliptical galaxies

• stabilisation of elliptical galaxies $\rightarrow$ velocity dispersion

• Jeans equations are 2 coupled nonlinear PDEs for the evolution of collisionless systems
  • first moment: continuity
    \[
    \frac{\partial \rho}{\partial t} + \text{div}(\rho \vec{u}) = 0 \quad (19)
    \]
  • second moment: momentum equation
    \[
    \frac{\partial \vec{u}}{\partial t} + \vec{u} \nabla \vec{u} = -\nabla \Phi - \text{div}(\rho \sigma^2) \quad (20)
    \]

• no viscosity, and velocity dispersion tensor \( \sigma_{ij}^2 = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle \) emulates (possibly anisotropic) pressure

• gravitational potential: self-consistently derived from Poisson’s equation \( \Delta \Phi = 4\pi G \rho \), closed system!

• in a virialised elliptical galaxy, \( \sigma_{ij} \) corresponds to \( \langle V \rangle \rightarrow \) stability
stability of spiral galaxies

- collisionless fluids can not build up pressure against gravity
- a rotating system can provide force balancing → centrifugal force
- spin-up: explained by tidal torquing
- spin-parameter $\lambda$

$$\lambda \equiv \frac{\omega}{\omega_0} = \frac{L/(MR^2)}{\sqrt{GM/R^3}} = \frac{L\sqrt{E}}{GM^{5/2}}$$

- specific angular momentum necessary for rotational support
- $\lambda \sim 1/2$ in spirals in $\Lambda$CDM cosmologies, rotation is the dominant supporting mechanism
SAURON observations of galaxies

source: SAURON experiment
**galaxy morphologies: 'tuning fork' diagramme**

*Edwin Hubble's Classification Scheme*

- **Ellipticals**: E0, E3, E5, E7, S0
- **Spirals**: Sa, Sb, Sc, SBa, SBb, SBC

*source: wikipedia*
galaxy clusters

- largest, gravitationally bound objects, with $M > M_*$
- quasar host structures at high redshift
- historically
  - visual identification (Abell catalogue)
  - need for dark matter: dynamical mass $\gg$ sum of galaxies (Zwicky)
- large clusters have masses of $10^{15} M_\odot / h$ and contain $\sim 10^3$ galaxies
X-ray emission of clusters

- the intra-cluster medium of clusters of galaxies is so hot ($T \simeq 10^7 \text{K}$) that it produces thermal X-ray radiation.
- the plasma is in hydrostatic equilibrium with gravity, therefore the density profile can be computed:

\[
\frac{dp}{dr} = -\frac{GM(r)}{r^2} \rho \rightarrow \frac{k_B T}{m} \frac{d\rho}{dr} + \frac{\rho k_B}{m} \frac{dT}{dr} = -\frac{GM}{r^2} \rho
\]  

(22)

for ideal gas with $p = \rho k_B T/m$.
- determination of mass: from measurement of the density and temperature profile:

\[
M(r) = -\frac{rk_B T}{Gm} \left(\frac{d \ln \rho}{d \ln r} + \frac{d \ln T}{d \ln r}\right)
\]  

(23)
X-ray emission of clusters: ROSAT data

- cluster is in hydrostatic equilibrium
- X-ray emissivity is $\propto \sqrt{T}\rho^2 \rightarrow$ fuzzy blobs
scaling relations

- virial relation allow the prediction of simple scaling relations
- valid for fully virialised systems, where the temperature reflects the release in gravitational binding energy
  - potential energy $\langle V \rangle \propto -GM^2/R$
  - size $M \propto R^3 \rightarrow \langle V \rangle \propto -M^{5/3}$
  - kinetic energy $\langle T \rangle \propto TM$
  - virial relation $2\langle T \rangle = -\langle V \rangle \rightarrow T \propto M^{2/3}$
  - X-ray luminosity $L_X \propto M^2 \sqrt{T}/R^3 \propto M^{4/3} \propto T^2$